M.Sc (Sem 3) Paper II.
Unit – 02

Weak Convergence:— Let X be a normed space X^* and X^{**} be the first and second dual spaces of X respectively.

1. A sequence $\{x_n\}$ in X is called weakly convergent in X.

In symbols $x_n \xrightarrow{w} x$, if there exists an element $x \in X$ such that $\lim_{n\to\infty} |f(x_n) - f(x)| = 0 \ \forall$ $f \in X^*$ ie for $\epsilon > 0$ there exists a natural number N such that $|f(x_n) - f(x)| \le \epsilon$ for $n > N$ and $\forall f \in X^*$.

2. A sequence $\{f_n\}$ in X^* is called weakly* convergent to f in X^* if $\lim_{n\to\infty} |f_n(x) - f(x)| = 0 \ \forall x \in X$.

Weak Cauchy sequence:— 1. A sequence $\{x_n\}$ in a normed space X is called a weak Cauchy sequence if $\{f(x_n)\}$ is a Cauchy sequence $\forall f \in X^*$.

2. A normed space X is called weakly complete if every weak Cauchy sequence of elements of X converges weakly to some other member of X.

Example (A) Show that weak convergence does not imply convergence to norm.

Solution:— By the application of the Riesz representation theorem to the Hilbert space $L_2(0, 2\pi)$, we find that

$$f(x) = \langle x, g \rangle = \int_0^{2\pi} x(t) g(t)\, dt \quad \text{—— (1)}$$

Consider the sequence $\{x_n(t)\}$ defined below

$$x_n(t) = \frac{\sin nt}{\pi} \quad \text{for } n = 1, 2, 3, \ldots$$

~~We know that~~ We show now that $\{x_n(t)\}$ is weakly convergent in $L_2(0, \pi)$ but is not norm convergent in $L_2(0, 2\pi)$.

From equation (1), we have

$$f(x_n) = \langle x_n, g \rangle = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin nt}{\pi}\, g(t)\, dt \quad (2)$$

The right-hand side of equation (2) is the trigonometric Fourier coefficient of $g(t) \in L_2(0,\pi)$.

By the Riemann-Lebesgue theorem concerning the behaviour of trigonometric Fourier coefficients

$$\frac{1}{\pi} \int_0^{2\pi} \sin nt\, g(t)\, dt \to 0 \text{ as } n \to \infty$$

or $f(x_n) \to 0$ as $n \to \infty$.

i.e. $f(x_n)$ converges weakly to 0.

We have

$$\|x_n - 0\| = \|x_n\| = \left(\int_0^{2\pi} |x_n(t)|^2 dt \right)^{1/2}$$

$$= \left(\int_0^{2\pi} \frac{|\sin nt|^2}{\pi^2}\, dt \right)^{1/2} = \frac{1}{\sqrt{\pi}}$$

Since $\frac{1}{\pi} \left(\int_0^{2\pi} \sin^2 nt\, dt \right)^{1/2} \neq 0 \;\forall n$, $\|x_n - 0\|$ cannot tend to 'zero' and therefore $\{x_n\}$ cannot converge in the norm.

Thus a weakly convergent sequence need not be convergent in the norm.

Ex-(B) Show that Weak* convergence does not imply Weak Convergence.

Solution:- Let $X = l_1$ (We know that l_1 is the dual of c_0 or m) and $Y = c_0$.

Then $Y^* \cong l_1$. Thus the dual of Y is X. Let $\{x^k\}$ be a sequence in X defined by the relation

$$x_j^k = 0 \quad \text{if } k \neq j$$
$$= 1 \quad \text{if } k = j$$

For $y = (y_1, y_2, y_3, y_k, \text{---}) \in c_0$. Let $x^k(y) = y_k$.

(x^k belongs to the dual of c_0. i.e. It is a bounded linear functional on c_0.)

Since $y \in c_0$, $\lim\limits_{k \to \infty} y_k = 0$ and so

$\lim\limits_{k \to \infty} x^k(y) = \lim\limits_{k \to \infty} y_k = 0$. Therefore, the sequence $\{x^k\}$ of the dual space of $Y = c_0$ Converges Weak* to zero, Now if $z \in X^* = l_\infty$ with $z = (z_1, z_2, \text{---})$ then $x^k(z) = z_k$. Since $z \in l_\infty$, $\{z_k\}$ is bounded with respect to k but need not converges to zero as $k \to \infty$. In fact

if $z = (1, 1, \text{---})$ then $x^k(z) \to 1$ as $k \to \infty$.

Therefore $\{x^k\}$ does not converge weakly.

This example shows that Weak* convergence does not imply weak convergence.

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Teacher's Signature